ANDREWS-GORDON TYPE IDENTITIES FROM COMBINATIONS OF VIRASORO CHARACTERS

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Abstract. For $p \in \{3, 4\}$ and all $p' > p$, with p' coprime to p, we obtain fermionic expressions for the combination $\chi^{p,p'}_{1,s} + q^{\Delta}\chi^{p,p'}_{p-1,s}$ of Virasoro (W_2) characters for various values of s, and particular choices of Δ. Equating these expressions with known product expressions, we obtain q-series identities which are akin to the Andrews-Gordon identities. For $p = 3$, these identities were conjectured by Bytsko. For $p = 4$, we obtain identities whose form is a variation on that of the $p = 3$ cases. These identities appear to be new.

The case $(p, p') = (3, 14)$ is particularly interesting because it relates not only to W_2, but also to W_3 characters, and offers W_3 analogues of the original Andrews-Gordon identities. Our fermionic expressions for these characters differ from those of Andrews *et al* which involve Gaussian polynomials.

1. Introduction

1.1. Identities and Virasoro Characters.

The q-series identities of Andrews–Gordon [1, 14] take the form:

(1)
$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i \,(\mathrm{mod}\, 2k+1)}}^{\infty} \frac{1}{1-q^n},$$

where $|q| < 1$ and, as usual, $(q)_0 = 1$ and $(q)_n = \prod_{i=1}^{n}(1-q^i)$ for $n > 0$. Here $k \geq 2$ and $1 \leq i \leq k$. The $k = 2$ cases are the famous Rogers-Ramanujan identities [17, 18].

In the past twenty years, it was recognised that (1) is an identity for the (normalised) character $\chi^{2,2k+1}_{1,i}$ of the minimal model $M(2, 2k + 1)_2$ of the Virasoro algebra. In fact, the left side of (1) has a combinatorial interpretation in terms of particles which are forbidden to overlap. It is for this reason that the left side of (1) is termed a *fermionic* expression.

The Virasoro minimal models $M(p, p')_2$ are labelled by coprime integers p and p' for which $1 < p < p'$.[1] They contain irreducible modules labelled by r and s with $1 \leq r < p$ and $1 \leq s < p'$. In [11, 16], the characters of these modules were calculated to be $\hat\chi^{p,p'}_{r,s} = q^{\Delta^{p,p'}_{r,s}}\chi^{p,p'}_{r,s}$, where the normalised

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[1] $M(p, p')_2$ is often denoted $\mathcal{M}(p, p')$ or $\mathcal{M}^{p,p'}$.

character $\chi^{p,p'}_{r,s}$ is given by:

$$\chi^{p,p'}_{r,s} = \frac{1}{(q)_\infty} \sum_{\lambda=-\infty}^{\infty} (q^{\lambda^2 pp' + \lambda(p'r-ps)} - q^{(\lambda p + r)(\lambda p' + s)}), \tag{2}$$

with $(q)_\infty = \prod_{i=1}^\infty (1-q^i)$, and the *conformal dimension* $\Delta^{p,p'}_{r,s}$ is given by:

$$\Delta^{p,p'}_{r,s} = \frac{(p'r-ps)^2 - (p'-p)^2}{4pp'}. \tag{3}$$

This expression (2) for $\chi^{p,p'}_{r,s}$ is of a different nature to those on either side of (1). It is known as a *bosonic* expression. For later purposes, it will be useful to note that $\chi^{p,p'}_{r,s} = \chi^{p,p'}_{p-r,p'-s}$ and that $\chi^{p,p'}_{r,s}|_{q=0} = 1$.

Following the recognition that (1) is an identity for $\chi^{2,2k+1}_{1,i}$, it was natural to seek fermionic expressions for other Virasoro characters. Gradually, as described in [6, 21], beginning with the pioneering work of the Stony Brook group [15], an increasingly wide range of characters were tackled in a series of works, culminating in [21] giving fermionic expressions for all minimal model Virasoro characters. In the simplest cases, such as the subset of the $p=3$ cases of $\chi^{p,p'}_{r,s}$ tackled by [2, 12], the fermionic expressions are strikingly similar to that in (1), with merely the coefficients of some of the parameters changed. However, expressions for other characters [15, 5], most notably the unitary characters $\chi^{p,p+1}_{r,s}$, necessitated the summand to have one or more Gaussian polynomial factors, where as usual, the Gaussian polynomial $\begin{bmatrix} P \\ N \end{bmatrix}_q$ is defined by:

$$\begin{bmatrix} P \\ N \end{bmatrix}_q = \begin{cases} \dfrac{(q)_P}{(q)_N (q)_{P-N}} & \text{if } 0 \le N \le P; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Such a multi-sum expression is known as a *fundamental fermionic form.* In [21], the fermionic expression for $\chi^{p,p'}_{r,s}$ is generally a sum over a number of fundamental fermionic forms.

Of course, to obtain an identity which is an analogue of (1), it is necessary to find an expression for the character which is a product similar in nature to that on the right. In fact, as explained in [9], $\chi^{p,p'}_{r,s}$ is a product of terms of the form $(1-q^n)^{-1}$ if and only if $p=2r$, $p'=2s$, $p=3r$ or $p'=3s$:

$$\chi^{2r,p'}_{r,s} = \prod_{\substack{n=1 \\ n\not\equiv 0, \pm rs \,(\mathrm{mod}\, rp')}}^{\infty} \frac{1}{1-q^n}, \qquad \chi^{p,2s}_{r,s} = \prod_{\substack{n=1 \\ n\not\equiv 0, \pm rs \,(\mathrm{mod}\, sp)}}^{\infty} \frac{1}{1-q^n}, \tag{5}$$

$$\chi^{3r,p'}_{r,s} = \prod_{\substack{n=1 \\ n\not\equiv 0, \pm rs \,(\mathrm{mod}\, 2rp') \\ n\not\equiv \pm 2r(p'-s) \,(\mathrm{mod}\, 4rp')}}^{\infty} \frac{1}{1-q^n}, \qquad \chi^{p,3s}_{r,s} = \prod_{\substack{n=1 \\ n\not\equiv 0, \pm rs \,(\mathrm{mod}\, 2sp) \\ n\not\equiv \pm 2s(p-r) \,(\mathrm{mod}\, 4sp)}}^{\infty} \frac{1}{1-q^n}. \tag{6}$$

These expressions are easily derived by applying Jacobi's triple product identity [13, eq. (II.28)] in the cases $p=2r$ and $p'=2s$, and Watson's quintuple product identity [13, ex. 5.6] in the cases $p=3r$ and $p'=3s$, to the expression (2). On identifying one of these product expressions with a

fermionic expression for the same $\chi^{p,p'}_{r,s}$, we obtain an identity which we refer to as a $M(p,p')_2$-identity.

1.2. **Combinations of Virasoro characters.** In [7], the sum and difference of certain pairs of characters were considered. Firstly, use of Watson's quintuple product identity and (2) leads to the following expression ([7, eqn. (2.26)]):

$$\chi^{3,p'}_{1,s} \pm q^{\frac{p'}{4}-\frac{s}{2}}\chi^{3,p'}_{2,s} = \frac{(\mp q^{\frac{p'}{4}-\frac{s}{2}}, \mp q^{\frac{p'}{4}+\frac{s}{2}}, q^{\frac{p'}{2}}; q^{\frac{p'}{2}})_\infty (q^s, q^{p'-s}; q^{p'})_\infty}{(q)_\infty}, \tag{7}$$

for $p' \not\equiv 0 \,(\mathrm{mod}\, 3)$ and $1 \le s < p'$, where $(a_1, a_2, \ldots, a_t; z)_\infty = \prod_{j=1}^t (a_j; z)_\infty$ with $(a;z)_\infty = \prod_{i=0}^\infty (1-az^i)$ as usual. It will be useful to note the following alternative form of this expression when $p' \neq 2s$:

$$\chi^{3,p'}_{1,s} \pm q^{\frac{p'}{4}-\frac{s}{2}}\chi^{3,p'}_{2,s} = \frac{(q^s, q^{\frac{p'}{2}-s}, q^{\frac{p'}{2}}; q^{\frac{p'}{2}})_\infty}{(q)_\infty (\pm q^{\frac{p'}{4}-\frac{s}{2}}, \pm q^{\frac{p'}{4}+\frac{s}{2}}; q^{\frac{p'}{2}})_\infty}. \tag{8}$$

Secondly, use of Jacobi's triple product identity and (2) leads to the following expression ([7, eqn. (2.22)]):

$$\chi^{4,p'}_{1,s} \pm q^{\frac{p'}{2}-s}\chi^{4,p'}_{3,s} = \frac{(q^s, \mp q^{\frac{p'}{2}-s}, \mp q^{\frac{p'}{2}}; \mp q^{\frac{p'}{2}})_\infty}{(q)_\infty}, \tag{9}$$

for $p' \not\equiv 0 \,(\mathrm{mod}\, 2)$ and $1 \le s < p'$. This may also be written:

$$\chi^{4,p'}_{1,s} \pm q^{\frac{p'}{2}-s}\chi^{4,p'}_{3,s} = \frac{(\mp q^{\frac{p'}{2}-s}, \mp q^{\frac{p'}{2}}, \mp q^{\frac{p'}{2}+s}, q^s, q^{p'-s}, q^{p'}; q^{p'})_\infty}{(q)_\infty}. \tag{10}$$

In what follows, we derive fermionic expressions for the '+' cases of the above character combinations. This leads to analogues of the Andrews-Gordon identities.

In recognition of its origin in a character combination, we refer to the identity for each sum $\chi^{3,p'}_{1,s} + q^{\frac{p'}{4}-\frac{s}{2}}\chi^{3,p'}_{2,s}$ of characters, as an $M(3,p')^+_2$-identity. Similarly, the identity for each sum $\chi^{4,p'}_{1,s} + q^{\frac{p'}{2}-s}\chi^{4,p'}_{3,s}$ of characters, is referred to as an $M(4,p')^+_2$-identity.

1.3. $\boldsymbol{M(3,7)_3}$**-identities.** For $n > 2$, the W_n algebra [22, 10] is a generalisation of the Virasoro algebra W_2. In [3], Andrews, Schilling and Warnaar applied a modified Bailey transform to obtain identities (referred to in [3] as Rogers-Ramanujan type identities) for three (of the possible four) characters of the minimal model $M(3,7)_3$ of W_3.[2] In Section 2.4 below, we list these identities together with a conjectured identity for the fourth character. Notably, the summand on the fermionic side of each of these identities contains a Gaussian polynomial.

In attempting to understand these $M(3,7)_3$-identities, we noticed that, since the W_n minimal model $M(p,p')_n$ has central charge

$$c^{p,p'}_n = (n-1)\left(1 - \frac{n(n+1)(p'-p)^2}{pp'}\right), \tag{11}$$

[2] One of these identities is also proved in [20] using Hall-Littlewood polynomials.

the W_3 minimal model $M(3,7)_3$ has the same central charge as that of the W_2 minimal model $M(3,14)_2$. This indicated that the characters of the $M(3,7)_3$ theory should be expressible as linear combinations of those of $M(3,14)_2$. In fact, the required combinations are precisely those of (7) for $s \in \{1,3,5\}$, together with the single character $\chi^{3,14}_{1,7}$. In each of the first three cases, the known fermionic expressions for the characters $\chi^{3,14}_{1,s}$ and $\chi^{3,14}_{2,s}$ each comprises a single fundamental fermionic form [6, 21], and sum conveniently to yield a single fundamental fermionic form. In the latter case, the known fermionic expression is a sum of two fundamental fermionic forms [21]. However, they sum conveniently to yield a single fundamental fermionic form in a manner that is similar to the other cases.

1.4. $\boldsymbol{M(p,p')^+_2}$**-identities.** Since the above means of combining W_2 characters extends readily to many $p=3$ and $p=4$ cases, the statement and proof of the $M(3,14)^+_2$-identities have been subsumed into the general case. The general $M(3,p')^+_2$-identities and $M(4,p')^+_2$-identities are stated in Sections 2.1 and 2.2 respectively. Their proofs are given in Section 3.

The $M(3,p')^+_2$-identities that we have obtained were conjectured by Bytsko [8]. Our $M(4,p')^+_2$-identities are (to the best of our knowledge) new.

As explained above, the $M(3,14)^+_2$-identities are also identities for the $M(3,7)_3$ characters. In order to compare our $M(3,14)^+_2$-identities with the identities of [3] listed in Section 2.4, we write out the former explicitly in Section 2.3. We see that we now have two different fermionic expressions for each of the four characters. In each case, the two fermionic expressions are not simply related to one another.

2. Andrews-Gordon type identities

In this section, we list our results. In each identity, if $s>k$ a sum of the form $(N_s+N_{s+1}+\cdots+N_k)$ is to be taken as 0. Each identity is proved in Section 3 using the procedure discussed above.

2.1. $\boldsymbol{M(3,p')^+_2}$**- and** $\boldsymbol{M(3,p')_2}$**-identities.** Theorems 2.1 and 2.3 below give $M(3,p')^+_2$-identities. Theorems 2.2 and 2.4 below give $M(3,p')_2$-identities. The $M(3,p')^+_2$-identities originate from the sum of two Virasoro characters, one that has $r=1$ and one that has $r=2$, as explained above. The $M(3,p')_2$-identities originate from one Virasoro character only, as can be seen from the proofs in Section 3.

In Theorems 2.1 and 2.3, the identities are indexed by the integers g and s. In these two cases, p' is related to g by $p'=3g+1$ and $p'=3g+2$ respectively. In Theorems 2.2 and 2.4, the identities are indexed by the integer h only. From h, one can determine both p' and s, as explained in Section 3.

Theorem 2.1. *If $g \geq 1$ and $1 \leq s \leq g+1$, then:*

(12)
$$\boxed{\sum_{\substack{N_1 \geq \cdots \geq N_{g-1} \geq 0 \\ M \geq 0}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+M) + \frac{g+1}{4}M^2 + \frac{g-s}{2}M + (N_s+N_{s+1}+\cdots N_{g-1})}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q)_M} = \frac{(-q^{\frac{3g+1}{4}-\frac{s}{2}}, -q^{\frac{3g+1}{4}+\frac{s}{2}}, q^{\frac{3g+1}{2}}; q^{\frac{3g+1}{2}})_\infty (q^s, q^{3g+1-s}; q^{3g+1})_\infty}{(q)_\infty}.}$$

Theorem 2.2. *Let $h \geq 1$. Then:*

$$\boxed{\sum_{\substack{N_1 \geq \cdots \geq N_{2h} \geq 0 \\ M \geq 0}} \frac{q^{\sum_{j=1}^{2h} N_j(N_j+M) + \frac{h+1}{2}M(M+1) - M + (N_h+N_{h+1}+\cdots+N_{2h})}}{(q)_{N_1-N_2}\cdots(q)_{N_{2h-1}-N_{2h}}(q)_{N_{2h}}(q)_M} = \frac{(q^{3h+2}; q^{3h+2})_\infty}{(q)_\infty}.} \tag{13}$$

Theorem 2.3. *If $g \geq 1$ and $1 \leq s \leq g+1$, then:*

(14)
$$\boxed{\sum_{\substack{N_1 \geq \cdots \geq N_{g-1} \geq 0 \\ M \geq 0}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+M) + \frac{g}{4}M^2 + \frac{g-s+1}{2}M + (N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q)_M} = \frac{(-q^{\frac{3g}{4}-\frac{s-1}{2}}, -q^{\frac{3g}{4}+\frac{s+1}{2}}, q^{\frac{3g}{2}+1}; q^{\frac{3g}{2}+1})_\infty (q^s, q^{3g+2-s}; q^{3g+2})_\infty}{(q)_\infty}.}$$

Theorem 2.4. *Let $h \geq 1$. Then:*

$$\boxed{\sum_{\substack{N_1 \geq \cdots \geq N_{2h-1} \geq 0 \\ M \geq 0}} \frac{q^{\sum_{j=1}^{2h-1} N_j(N_j+M) + \frac{h}{2}M(M+1) + (N_h+N_{h+1}+\cdots+N_{2h-1})}}{(q)_{N_1-N_2}\cdots(q)_{N_{2h-2}-N_{2h-1}}(q)_{N_{2h-1}}(q)_M} = \frac{(q^{3h+1}; q^{3h+1})_\infty}{(q)_\infty}.} \tag{15}$$

We note that, in view of the equality between the right sides of (7) and (8), in the cases in which both p' and $p'/2 - s$ are even, the identities do not involve fractional powers of q, with the right side readily written as a product of terms of the form $(1-q^n)^{-1}$ for various $n \in \mathbb{Z}_{>0}$. Similarly, in the cases in which p' even and $p'/2 - s$ is odd, the right side is readily written as a product of terms of the form $(1 - q^{\frac{n}{2}})^{-1}$ for various $n \in \mathbb{Z}_{>0}$, and the identities may be naturally viewed as identities in the indeterminate $q^{1/2}$,

When p' is odd, the identities may be naturally viewed as identities in the indeterminate $q^{1/4}$, with in particular, the product expression able to be readily written as a product of terms of the forms $(1-q^{\frac{n}{4}})^{-1}$ and $(1+q^{\frac{m}{2}})^{-1}$ for various $n, m \in \mathbb{Z}_{>0}$.

The identities of Theorems 2.2 and 2.4, which are indexed by the integer h, do not involve fractional powers of q. They are $M(3, 6h+4)_2$- and $M(3, 6h+2)_2$-identities respectively.

2.2. $\boldsymbol{M(4,p')_2^+}$**-identities.** Theorems 2.5 and 2.6 below give $M(4,4g+1)_2^+$-identities, and Theorems 2.7 and 2.8 below give $M(4,4g+3)_2^+$-identities.

Theorem 2.5. *Let $g \geq 1$ and $1 \leq s \leq g+1$. Then:*

(16)
$$\boxed{\begin{aligned}\sum_{\substack{N_1\geq\cdots\geq N_{g-1}\geq 0\\ M\geq 0}} &\frac{q^{\sum_{j=1}^{g-1} N_j(N_j+2m)+(g+\frac{1}{2})M^2+(g-s)M+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q^{\frac{1}{2}};q)_M(q^2;q^2)_M}\\ &= \frac{(-q^{2g-s+\frac{1}{2}},-q^{2g+\frac{1}{2}},-q^{2g+s+\frac{1}{2}},q^s,q^{4g+1-s},q^{4g+1};q^{4g+1})_\infty}{(q)_\infty}.\end{aligned}}$$

Theorem 2.6. *Let $g \geq 1$. Then:*

(17)
$$\boxed{\begin{aligned}\sum_{\substack{N_1\geq\cdots\geq N_{g-1}\geq 0\\ M\geq 0}} &\frac{q^{\sum_{j=1}^{g-1} N_j(N_j+2M+1)+(g+\frac{1}{2})M^2+gM}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q^{\frac{1}{2}};q)_{M+1}(q^2;q^2)_M}\\ &= \frac{(-q^{\frac{1}{2}},-q^{2g+\frac{1}{2}},-q^{4g+\frac{1}{2}},q^{2g},q^{2g+1},q^{4g+1};q^{4g+1})_\infty}{(q)_\infty}.\end{aligned}}$$

Theorem 2.7. *Let $g \geq 1$ and $1 \leq s \leq g+1$. Then:*

(18)
$$\boxed{\begin{aligned}\sum_{\substack{N_1\geq\cdots\geq N_{g-1}\geq 0\\ M\geq 0}} &\frac{q^{\sum_{j=1}^{g-1} N_j(N_j+2M)+M(Mg+g+1-s)+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q^{\frac{1}{2}};q)_M(q^2;q^2)_M}\\ &= \frac{(-q^{2g-s+\frac{3}{2}},-q^{2g+\frac{3}{2}},-q^{2g+s+\frac{3}{2}},q^s,q^{4g+3-s},q^{4g+3};q^{4g+3})_\infty}{(q)_\infty}.\end{aligned}}$$

Theorem 2.8. *Let $g \geq 1$. Then:*

(19)
$$\boxed{\begin{aligned}\sum_{\substack{N_1\geq\cdots\geq N_{g-1}\geq 0\\ M\geq 0}} &\frac{q^{\sum_{j=1}^{g-1} N_j(N_j+2M+1)+gM(M+1)}}{(q)_{N_1-N_2}\cdots(q)_{N_{g-2}-N_{g-1}}(q)_{N_{g-1}}(q^{\frac{1}{2}};q)_{M+1}(q^2;q^2)_M}\\ &= \frac{(-q^{\frac{1}{2}},-q^{2g+\frac{3}{2}},-q^{4g+\frac{5}{2}},q^{2g+1},q^{2g+2},q^{4g+3};q^{4g+3})_\infty}{(q)_\infty}.\end{aligned}}$$

In each case above, the identities may be viewed as identities in the indeterminate $q^{1/2}$, with the product expression readily written as a product of terms of the forms $(1-q^{\frac{n}{2}})^{-1}$ and $(1+q^m)^{-1}$ for various $n,m \in \mathbb{Z}_{>0}$.

2.3. $\boldsymbol{M(3,7)_3}$ **Andrews-Gordon identities.** The $g=4$ cases of Theorem 2.3 for $s \in \{1,3,5\}$ and the $h=2$ case of Theorem 2.4, now yield identities for the four $M(3,7)_3$ characters. Written out in full, these are:

(20a)
$$\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+{n_3}^2+{n_4}^2+(n_1+2n_2+3n_3)n_4+(n_1+2n_2+3n_3)+2n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-2})(1-q^{7n-3})^2(1-q^{7n-4})^2(1-q^{7n-5})},$$

(20b)
$$\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3^2+n_4^2+(n_1+2n_2+3n_3)n_4+n_3+n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})(1-q^{7n-2})^2(1-q^{7n-5})^2(1-q^{7n-6})},$$

(20c)
$$\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3^2+n_4^2+(n_1+2n_2+3n_3)n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2},$$

(20d)
$$\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3^2+n_4^2+(n_1+2n_2+3n_3)n_4+(n_2+2n_3)+n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})(1-q^{7n-2})(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-5})(1-q^{7n-6})}.$$

2.4. **$M(3,7)_3$ ASW identities.** In [3], Andrews, Schilling and Warnaar obtained q-series identities for three characters of the $M(3,7)_3$ minimal model. These identities differ from those we have derived above. For comparison purposes, we list these identities here, together with a conjectured identity (21b) for the fourth $M(3,7)_3$ character:

(21a)
$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2-n_1n_2+n_2^2+n_1+n_2}}{(q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}_q$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-2})(1-q^{7n-3})^2(1-q^{7n-4})^2(1-q^{7n-5})},$$

(21b)
$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2-n_1n_2+n_2^2+n_2}}{(q)_{n_1}} \begin{bmatrix} 2n_1+1 \\ n_2 \end{bmatrix}_q$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})(1-q^{7n-2})^2(1-q^{7n-5})^2(1-q^{7n-6})},$$

$$\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2-n_1n_2+n_2^2}}{(q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix}_q = \prod_{n=1}^{\infty}\frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2}, \tag{21c}$$

$$\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2-n_1n_2+n_2^2+n_1}}{(q)_{n_1}}\begin{bmatrix}2n_1+1\\ n_2\end{bmatrix}_q = \sum_{n_1,n_2\geq 0}\frac{q^{n_1^2-n_1n_2+n_2^2+n_2}}{(q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix}_q \tag{21d}$$
$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^{7n-1})(1-q^{7n-2})(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-5})(1-q^{7n-6})}.$$

2.5. **Special cases.** Here we note that the simplest cases of the results in Section 2 yield known identities of Rogers-Ramanujan type.

The $g=1$ cases of Theorem 2.1 yield the $z=q^{\frac{1}{2}}$ and $z=1$ specialisations of Euler's formula [13, II.2].

The $g=1$ cases of Theorem 2.3 yield the following identities after substituting $q\to q^4$:

$$\sum_{m=0}^{\infty}\frac{q^{m(m+2)}}{(q^4;q^4)_m}=\frac{(-q^3,-q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q^4;q^4)_\infty}, \tag{22a}$$

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^4;q^4)_m}=\frac{(-q,-q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q^4;q^4)_\infty}. \tag{22b}$$

These identities are equivalent to identities of Rogers [17]. They also appear in [19, eqns. (16) & (20)].

The $g=1$ cases of Theorems 2.5 and 2.6 yield the following identities after substituting $q\to q^2$:

$$\sum_{m=0}^{\infty}\frac{q^{3m^2}}{(q;q^2)_m(q^4;q^4)_m}=\frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}, \tag{23a}$$

$$\sum_{m=0}^{\infty}\frac{q^{3m^2+2m}}{(q;q^2)_{m+1}(q^4;q^4)_m}=\sum_{m=0}^{\infty}\frac{q^{3m^2-2m}}{(q;q^2)_m(q^4;q^4)_m}=\frac{(-q,-q^5,-q^9;q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}. \tag{23b}$$

In an alternative form, (23a) and the second identity in (23b) are due to Rogers [17]. They also appear in [19, eqns. (19) & (15)]. Bailey [4] also derives (23a) and the two identities (23b).

From Theorems 2.7 and 2.8, we obtain:

$$\sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q;q^2)_m(q^4;q^4)_m}=\frac{(-q^5,-q^7,-q^9;q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty}, \tag{24a}$$

$$\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q^2)_m(q^4;q^4)_m}=\frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty}, \tag{24b}$$

$$\sum_{m=0}^{\infty} \frac{q^{2m(m+1)}}{(q;q^2)_{m+1}(q^4;q^4)_m} = \frac{(-q,-q^7,-q^{13};q^{14})_\infty}{(q^2,q^4,q^{10},q^{12};q^{14})_\infty}. \tag{24c}$$

These identities are alternative forms of the Rogers-Selberg identities which are originally due to Rogers [17]. They also appear in [19, eqns. (32), (33) & (31)].

3. Proofs of fermionic combinations

In this section, we prove the expressions of Section 2. In each case, the first step is to extract fermionic expressions for characters $\chi^{3,p'}_{1,s}$ or $\chi^{4,p'}_{1,s}$ from [21]. Some of these expressions can also be found in [5, 6]. Having obtained the fermionic forms, the required proofs result upon combining them appropriately and then using either (7) and (9).

In each of the cases below, we use the notation of [21].

3.1. **The** $(p,p') = (3, 3g+1)$ **case.** The continued fraction of $p'/p = (3g+1)/3$ is $[g,3]$. Then, from [21, §1.3], $n=1$, $t=g+1$ and $t_1 = g-1$. The set $\mathcal{T}$ of Takahashi lengths is given by $\mathcal{T} = (1,2,\ldots,g,g+1)$. The set $\tilde{\mathcal{T}}$ of truncated Takahashi lengths is given by $\tilde{\mathcal{T}} = (1)$. From [21, §1.11], $\boldsymbol{B}$ is the $(g-1)\times(g-1)$ matrix $\boldsymbol{B} = \{B_{j\ell}\}_{1\le j,\ell\le g-1}$ with entries $B_{j\ell} = \min\{j,\ell\}$, and $\overline{\boldsymbol{C}}$ is the 1×1 matrix (2). The matrix $\overline{\boldsymbol{C}}^*$ is the 1×1 matrix (-1).

Throughout this subsection, we take $N_j = n_j + n_{j+1} + \cdots + n_{g-1}$ for $1 \le j \le g-1$, and the $(g+1)$-dimensional vectors $\boldsymbol{e}_j$ are defined by $\boldsymbol{e}_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{g+1,j})$ for $0 \le j \le g+2$.

Lemma 3.1. *Let $g \ge 1$ and $1 \le s \le g+1$. Then:*

$$\chi^{3,3g+1}_{1,s} = \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\ge0}\\ m\in\mathbb{Z}_{\ge0}\\ m\equiv 0 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m) + \frac{g+1}{4}m^2 + \frac{g-s}{2}m + (N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_m} \tag{25a}$$

and

$$\chi^{3,3g+1}_{2,s} = q^{-\frac{3g}{4}+\frac{s}{2}-\frac14} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\ge0}\\ m\in\mathbb{Z}_{\ge0}\\ m\equiv 1 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m) + \frac{g+1}{4}m^2 + \frac{g-s}{2}m + (N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_m}. \tag{25b}$$

Each term $(N_s + N_{s+1} + \cdots + N_{g-1})$ is to be taken as 0 whenever $s \ge g$.

Proof: Since with $1 \le s \le g+1$, we have $s \in \mathcal{T}$, and for $r=1$, we have $r \in \tilde{\mathcal{T}}$, the fermionic form [21, (1.16)] for $\chi^{p,p'}_{r,s}$ comprises a single fundamental form [21, (1.17)]. The summation in [21, (1.17)] is then over integers $n_1, n_2, \ldots, n_{g-1}$ and m (this last parameter is named m_g in [21, (1.17)]), where the parity of m is restricted.

From [21, §1.7], we obtain:

$$\boldsymbol{u}^L = \begin{cases} \boldsymbol{e}_g & \text{if } s = g+1; \\ 0 & \text{if } s = g; \\ \boldsymbol{e}_{s-1} - \boldsymbol{e}_{g-1} & \text{if } s \le g-1. \end{cases}$$

From [21, §1.9], we then get $\overline{\boldsymbol{u}}^L_\flat = (1)$ if $s = g+1$ and $\overline{\boldsymbol{u}}^L_\flat = (0)$ if $s \le g$. On the other hand, $\boldsymbol{u}^R = \boldsymbol{e}_g$ and $\overline{\boldsymbol{u}}^R_\sharp = (0)$. For [21, (1.18)], we then get $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{g-1})$ where, if $s \in \{g, g+1\}$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m$ and $\tilde{n}_j = n_j$ for $1 \le j < g-1$, and if $1 \le s < g$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m + \frac{1}{2}$, $\tilde{n}_{s-1} = n_{s-1} - \frac{1}{2}$ and $\tilde{n}_j = n_j$ for $1 \le j < s-1$ and $s \le j < g-1$. The expression [21, (1.17)] then yields equation (25a) up to an overall factor, where m is summed over even integers because here $\overline{\boldsymbol{u}} = (0)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (0)$. The overall factor may be obtained after calculating γ as in [21, §1.10]. However, we can bypass this calculation by noting that the smallest value of the exponent in the numerator of the summand in equation (27a) occurs when $n_1 = n_2 = \cdots = n_{2g-1} = m = 0$. Since $\chi^{p,p'}_{r,s}|_{q=0} = 1$, it follows that the required factor is simply 1.

For the $\chi^{3,p'}_{2,s}$ case, we obtain $\boldsymbol{u}^L$ as above. Using $r = 2$ instead of $r = 1$ results in $\boldsymbol{u}^R = \boldsymbol{e}_g + \boldsymbol{e}_{g+1}$. This yields $\overline{\boldsymbol{u}}^R_\sharp = (0)$ as above. However here, $m \equiv 1 \pmod 2$ in the summation because $\overline{\boldsymbol{u}} = (1)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) \equiv (-1)$. Thus, apart from the overall factor, the fermionic form [21, (1.17)] for $\chi^{3,p'}_{2,s}$ differs from the $r = 1$ case above only in that m is summed over odd integers. The smallest value of the exponent in the numerator of the summand in equation (25b) occurs when $n_1 = n_2 = \cdots = n_{2g-1} = 0$ and $m = 1$. It follows that the required factor is $q^{-\frac{3g}{4} + \frac{s}{2} - \frac{1}{4}}$. □

Proof of Theorem 2.1: Combining Lemma 3.1 with equation (7) immediately gives the required result. □

Lemma 3.2. *Let $h \ge 1$ and set $s = 3h+2$. Then:*

$$\chi^{3,2s}_{1,s} = \sum_{\substack{n_1, n_2, \ldots, n_{2h} \in \mathbb{Z}_{\ge 0} \\ m \in \mathbb{Z}_{\ge 0}}} \frac{q^{\sum_{j=1}^{2h} N_j(N_j+m) + \frac{h+1}{2} m(m+1) - m + (N_h + N_{h+1} + \cdots + N_{2h})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{2h}}(q)_m}, \tag{26}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{2h}$.

Proof: Here $p' = 6h+4$, and so the above data applies for $g = 2h+1$. Since $s \notin \mathcal{T}$, the fermionic expression [21, (1.16)] is a sum over more than one fundamental fermionic form. In fact, the Takahashi tree for s ([21, §1.6]) has precisely two leaf nodes. Since the truncated Takahashi tree for $r = 1$ has one leaf node, the sum [21, (1.16)] is over two terms.

For the leaf node a_{00} of the Takahashi tree for s, we obtain ([21, §1.7]) $d = 2$, $\Delta_1 = 1$, $\sigma_1 = g$, $\tau_1 = g+2$, $\Delta_2 = -1$, $\tau_2 = g-2$ and $\sigma_2 = (g-3)/2$. This leads to $\boldsymbol{u}^L = \boldsymbol{e}_{(g-3)/2} - \boldsymbol{e}_{g-2} + \boldsymbol{e}_g + \boldsymbol{e}_{g+1}$. Thereupon, $\overline{\boldsymbol{u}}^L_\flat = (1)$. On the other hand $r = 1$ leads to $\boldsymbol{u}^R = \boldsymbol{e}_g$ and $\overline{\boldsymbol{u}}^R_\sharp = (0)$. For this case, the calculation described in ([21, §1.10]) yields the constant term

$\gamma(\chi^L,\chi^R) = -(g-1)/2$. The fundamental fermionic form [21, (1.17)] for this case is then given by precisely the right side of equation (26), with the summation restricted to odd integers m because $\overline{\boldsymbol{u}} = (1)$ here, which yields $\overline{\boldsymbol{Q}}(\boldsymbol{u}) \equiv (-1)$.

For the leaf node a_{10} of the Takahashi tree for s, we obtain $d = 2$, $\Delta_1 = -1$, $\sigma_1 = g-1$, $\tau_1 = g+2$, $\Delta_2 = 1$, $\tau_2 = g-2$ and $\sigma_2 = (g-3)/2$. This leads to $\boldsymbol{u}^L = \boldsymbol{e}_{(g-3)/2} - \boldsymbol{e}_{g-2} + \boldsymbol{e}_g$. Thereupon, $\overline{\boldsymbol{u}}^L_\flat = (1)$. Again we then obtain $\gamma(\chi^L,\chi^R) = -(g-1)/2$. The fundamental fermionic form for this case is then given by precisely the right side of equation (26), with the summation restricted to even integers m because $\overline{\boldsymbol{u}} = (0)$ here, which yields $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (0)$. The expression (26) follows. □

Proof of Theorem 2.2: Combining Lemma 3.2 with the $p' = 2s$ case of equation (7), and cancelling a factor of 2 from each side, yields the required result. □

3.2. **The $(p,p') = (3,3g+2)$ case.** The continued fraction of $p'/p = (3g+2)/3$ is $[g,1,2]$. Then, from [21, §1.3], $n = 2$, $t = g+1$, $t_1 = g-1$ and $t_2 = g$. The set $\mathcal{T}$ of Takahashi lengths is given by $\mathcal{T} = (1,2,\ldots,g+1)$. The set $\tilde{\mathcal{T}}$ of truncated Takahashi lengths is given by $\tilde{\mathcal{T}} = (1)$. From [21, §1.11], $\boldsymbol{B}$ is the $(g-1)\times(g-1)$ matrix $\boldsymbol{B} = \{B_{j\ell}\}_{1\le j,\ell\le g-1}$ with entries $B_{j\ell} = \min\{j,\ell\}$, and $\overline{\boldsymbol{C}}$ is the 1×1 matrix (1). The matrix $\overline{\boldsymbol{C}}^*$ is the 1×1 matrix (-1).

Throughout this subsection, we take $N_j = n_j + n_{j+1} + \cdots + n_{g-1}$ for $1 \le j \le g-1$, and the $(g+1)$-dimensional vectors $\boldsymbol{e}_j$ are defined by $\boldsymbol{e}_j = (\delta_{1j},\delta_{2j},\ldots,\delta_{g+1,j})$ for $0 \le j \le g+2$.

Lemma 3.3. *Let $g \ge 1$ and $1 \le s \le g+1$. Then:*

$$\chi^{3,3g+2}_{1,s} = \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\ge0}\\ m\in\mathbb{Z}_{\ge0}\\ m\equiv0\,(\mathrm{mod}\,2)}} \frac{q^{\sum_{j=1}^{g-1}N_j(N_j+m)+\frac{g}{4}m^2+\frac{g-s+1}{2}m+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_m} \tag{27a}$$

and

(27b)

$$\chi^{3,3g+2}_{2,s} = q^{-\frac{3g}{4}+\frac{s}{2}-\frac12} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\ge0}\\ m\in\mathbb{Z}_{\ge0}\\ m\equiv1\,(\mathrm{mod}\,2)}} \frac{q^{\sum_{j=1}^{g-1}N_j(N_j+m)+\frac{g}{4}m^2+\frac{g-s+1}{2}m+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_m}.$$

Each term $(N_s+N_{s+1}+\cdots+N_{g-1})$ is to be taken as 0 whenever $s \ge g$.

Proof: Since the expressions here are obtained in a manner very similar to those of Lemma 3.1, we do not give the details. We only note that with $1 \le s \le g+1$ here, $s \in \mathcal{T}$ and then:

$$\boldsymbol{u}^L = \begin{cases} 0 & \text{if } s = g+1;\\ -\boldsymbol{e}_g & \text{if } s = g;\\ \boldsymbol{e}_{s-1} - \boldsymbol{e}_{g-1} - \boldsymbol{e}_g & \text{if } s \le g-1. \end{cases}$$

□

Proof of Theorem 2.3: Combining Lemma 3.3 with equation (7) immediately yields the required result. □

Lemma 3.4. *Let $h \ge 1$ and $s = 3h+1$. Then:*

$$\chi^{3,2s}_{1,s} = \sum_{\substack{n_1,n_2,\ldots,n_{2h-1}\in\mathbb{Z}_{\ge 0}\\ m\in\mathbb{Z}_{\ge 0}}} \frac{q^{\sum_{j=1}^{2h-1} N_j(N_j+m)+\frac{h}{2}m(m+1)+(N_h+N_{h+1}+\cdots+N_{2h-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{2h-1}}(q)_m}, \tag{28}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{2h-1}$.

Proof: Since the expression here is obtained in a manner very similar to that of Lemma 3.2, we do not give the details. We only note that, after setting $g = 2h$, the leaf node a_{00} of the Takahashi tree for s yields $d = 2$, $\Delta_1 = 1$, $\sigma_1 = g$, $\tau_1 = g+2$, $\Delta_2 = -1$, $\tau_2 = g-1$ and $\sigma_2 = g/2-1$, which leads to $\boldsymbol{u}^L = \boldsymbol{e}_{g/2-1} - \boldsymbol{e}_{g-1} + \boldsymbol{e}_{g+1}$, $\overline{\boldsymbol{u}}^L_\flat = (0)$ and $\gamma(\chi^L,\chi^R) = -g/2$; and the leaf node a_{10} yields $d = 2$, $\Delta_1 = -1$, $\sigma_1 = g+2$, $\tau_1 = g+2$, $\Delta_2 = 1$, $\tau_2 = g-1$ and $\sigma_2 = g/2-1$, which leads to $\boldsymbol{u}^L = \boldsymbol{e}_{g/2-1} - \boldsymbol{e}_{g-1}$, $\overline{\boldsymbol{u}}^L_\flat = (0)$ and $\gamma(\chi^L,\chi^R) = -g/2$. □

Proof of Theorem 2.4: Combining Lemma 3.4 with the $p' = 2s$ case of equation (7), and cancelling a factor of 2 from each side, yields the required result. □

3.3. **The $(p,p') = (4,4g+1)$ case.** In this and the following subsection, we concentrate on the cases where $p = 4$. In these cases, we make use of the following simple consequences of the q-binomial theorem [13, II.4]:

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k^2} \begin{bmatrix} P \\ k \end{bmatrix}_q = (-q^{\frac{1}{2}};q)_P, \tag{29a}$$

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k^2-Pk} \begin{bmatrix} P \\ k \end{bmatrix}_q = q^{-\frac{1}{2}P^2}(-q^{\frac{1}{2}};q)_P. \tag{29b}$$

The continued fraction of $p'/p = (4g+1)/4$ is $[g,4]$. Then, from [21, §1.3], $n = 1$, $t = g+2$ and $t_1 = g-1$. The set $\mathcal{T}$ of Takahashi lengths is given by $\mathcal{T} = (1,2,\ldots,g,g+1,2g+1)$. The set $\tilde{\mathcal{T}}$ of truncated Takahashi lengths is given by $\tilde{\mathcal{T}} = (1,2)$. From [21, §1.11], $\boldsymbol{B}$ is the $(g-1)\times(g-1)$ matrix $\boldsymbol{B} = \{B_{j\ell}\}_{1\le j,\ell\le g-1}$ with entries $B_{j\ell} = \min\{j,\ell\}$, and $\overline{\boldsymbol{C}}$ is the 2×2 matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. The matrix $\overline{\boldsymbol{C}}^*$ is the 2×2 matrix $\left(\begin{smallmatrix} -1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$.

Throughout this subsection, we take $N_j = n_j + n_{j+1} + \cdots + n_{g-1}$ for $1 \le j \le g-1$, and the $(g+2)$-dimensional vectors $\boldsymbol{e}_j$ are defined by $\boldsymbol{e}_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{g+2,j})$ for $0 \le j \le g+3$.

Lemma 3.5. *Let $g \geq 1$ and $1 \leq s \leq g+1$. Then:*

$$\chi^{4,4g+1}_{1,s} = \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\equiv 0\,(\mathrm{mod}\,2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g+1}{4}m_1^2+\frac{1}{2}m_2^2-\frac{1}{2}m_1m_2+\frac{1}{2}(g-s)m_1+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}m_1 \\ m_2 \end{bmatrix}_q \tag{30a}$$

and

$$\chi^{4,4g+1}_{3,s} = q^{s-2g-\frac{1}{2}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1\equiv 0\,(\mathrm{mod}\,2)\\ m_2\equiv 1\,(\mathrm{mod}\,2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g+1}{4}m_1^2+\frac{1}{2}m_2^2-\frac{1}{2}m_1m_2+\frac{1}{2}(g-s)m_1+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}m_1 \\ m_2 \end{bmatrix}_q . \tag{30b}$$

Each term $(N_s+N_{s+1}+\cdots+N_{g-1})$ is to be taken as 0 whenever $s \geq g$.

Proof: Since with $s \leq g+1$ and $r=1$, we have $s \in \mathcal{T}$ and $r \in \tilde{\mathcal{T}}$, the fermionic expression [21, (1.16)] for $\chi^{p,p'}_{r,s}$ comprises a single fundamental fermionic form [21, (1.17)]. The summation in [21, (1.17)] is then over integers $n_1, n_2, \ldots, n_{g-1}$ and m_1 and m_2 (these latter two parameters are named m_g and m_{g+1} in [21, (1.17)]), with the parity of m_1 and m_2 are restricted.

From [21, §1.7], we obtain:

$$\boldsymbol{u}^L = \begin{cases} \boldsymbol{e}_g & \text{if } s = g+1; \\ 0 & \text{if } s = g; \\ \boldsymbol{e}_{s-1} - \boldsymbol{e}_{g-1} & \text{if } s < g. \end{cases}$$

From [21, §1.9], we then get $\overline{\boldsymbol{u}}^L_\flat = (1,0)$ if $s = g+1$ and $\overline{\boldsymbol{u}}^L_\flat = (0,0)$ if $s \leq g$. On the other hand, $\boldsymbol{u}^R = \boldsymbol{e}_g$ and $\overline{\boldsymbol{u}}^R_\sharp = (0,0)$. For [21, (1.18)], we then get $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{g-1})$ where, if $s \in \{g, g+1\}$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m_1$ and $\tilde{n}_j = n_j$ for $1 \leq j < g-1$, and if $1 \leq s < g$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m_1 + \frac{1}{2}$, $\tilde{n}_{s-1} = n_{s-1} - \frac{1}{2}$, and $\tilde{n}_j = n_j$ for $1 \leq j < s-1$ and $s \leq j < g-1$. The expression [21, (1.17)] then yields equation (30a) up to an overall factor, where m_1 and m_2 are each summed over even integers because here $\overline{\boldsymbol{u}} = (0,0)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (0,0)$. The overall factor may be obtained by noting that the smallest value of the exponent in the numerator of the summand in equation (30a) occurs when $n_1 = n_2 = \cdots = n_{g-1} = m_1 = m_2 = 0$. Since $\chi^{p,p'}_{r,s}|_{q=0} = 1$, it follows that the required factor is simply 1.

For the second expression, we obtain $\boldsymbol{u}^L$ as above. Using $r=3$ instead of $r=1$ results in $\boldsymbol{u}^R = \boldsymbol{e}_g + \boldsymbol{e}_{g+2}$. This yields $\overline{\boldsymbol{u}}^R_\sharp = (0,0)$ as above. However here, $m_1 \equiv 0\,(\mathrm{mod}\,2)$ and $m_2 \equiv 1\,(\mathrm{mod}\,2)$ in the summation because $\overline{\boldsymbol{u}} = (0,1)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) \equiv (-2,-1)$. Thus, apart from the overall factor, the sum form [21, (1.17)] for $\chi^{4,4g+1}_{3,s}$ differs from the $r=1$ case above only in that m_2 is summed over odd integers. The smallest value of the exponent

in the numerator of the summand in equation (30b) occurs when $n_1 = n_2 = \cdots = n_{2g-1} = 0$ and $m_1 = 2$ and $m_2 = 1$. It follows that the required factor is $q^{s-2g-\frac{1}{2}}$. $\square$

Proof of Theorem 2.5: The expressions (30a) and (30b) of Lemma 3.5 may be combined to yield a sum over all $m_2 \in \mathbb{Z}_{\geq 0}$. On performing this summation using (29b) with $P = \frac{1}{2}m_1$, we obtain:

$$\chi^{4,4g+1}_{1,s} + q^{2g+\frac{1}{2}-s}\chi^{4,4g+1}_{3,s} = \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1\in\mathbb{Z}_{\geq 0}\\ m_1\equiv 0 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{2g+1}{8}m_1^2+\frac{1}{2}(g-s)m_1+(N_s+\cdots+N_{g-1})}(-q^{\frac{1}{2}};q)_{\frac{1}{2}m_1}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}}.$$

On setting $m_1 = 2M$ and noting that $(q)_{2M} = (q;q^2)_M(q^2;q^2)_M = (-q^{\frac{1}{2}};q)_M$ $(q^{\frac{1}{2}};q)_M(q^2;q^2)_M$, we obtain the left side of (16). The right side follows from (10). $\square$

Lemma 3.6. *Let $g \geq 1$. Then:*
(31a)

$$\chi^{4,4g+1}_{1,2g+1} = q^{-\frac{g+1}{4}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1\equiv 1 \,(\mathrm{mod}\, 2)\\ m_2\equiv 0 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g+1}{4}m_1^2+\frac{1}{2}m_2^2-\frac{1}{2}m_1m_2-\frac{1}{2}m_2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1+1) \\ m_2 \end{bmatrix}_q$$

and
(31b)

$$\chi^{4,4g+1}_{3,2g+1} = q^{-\frac{g-1}{4}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\equiv 1 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g+1}{4}m_1^2+\frac{1}{2}m_2^2-\frac{1}{2}m_1m_2-\frac{1}{2}m_2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1+1) \\ m_2 \end{bmatrix}_q.$$

Proof: Here $s = 2g+1$ so that $s \in \mathcal{T}$. Thereupon, $\boldsymbol{u}^L = \boldsymbol{e}_{g+1}$, $\overline{\boldsymbol{u}}^L_\flat = (0,1)$ and $\tilde{\boldsymbol{n}} = (n_1, n_2, \ldots, n_{g-2}, n_{g-1} + \frac{1}{2}m_1)$. The proof now follows the lines of Lemma 3.5, noting that for $r = 1$, we obtain $\boldsymbol{u} = (1,0)$ whereupon $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (1,0)$; and for $r = 3$, we obtain $\boldsymbol{u} = (1,1)$ whereupon $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (1,1)$. $\square$

Proof of Theorem 2.6: After noting that $\chi^{4,4g+1}_{1,2g+1} = \chi^{4,4g+1}_{3,2g}$ and $\chi^{4,4g+1}_{3,2g+1} = \chi^{4,4g+1}_{1,2g}$, expressions (31a) and (31b) of Lemma 3.6 may be combined to yield a sum over all $m_2 \in \mathbb{Z}_{\geq 0}$. On performing this summation using (29b) with $P = \frac{1}{2}(m_1+1)$, we obtain:

$$\chi^{4,4g+1}_{1,2g} + q^{\frac{1}{2}}\chi^{4,4g+1}_{3,2g} = q^{-\frac{2g-1}{8}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1\in\mathbb{Z}_{\geq 0}\\ m_1\equiv 1 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{2g+1}{8}m_1^2-\frac{1}{4}m_1}(-q^{\frac{1}{2}};q)_{\frac{1}{2}(m_1+1)}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}}.$$

On setting $m_1 = 2M+1$ and noting $(q)_{2M+1} = (q;q^2)_{M+1}(q^2;q^2)_M = (-q^{\frac{1}{2}};q)_{M+1}\,(q^{\frac{1}{2}};q)_{M+1}(q^2;q^2)_M$, we obtain the left side of (17). The right side follows from (10). □

3.4. **The** $(p,p')=(4,4g+3)$ **case.** The continued fraction of $p'/p=(4g+3)/4$ is $[g,1,3]$. Then, from [21, §1.3], $n=1$, $t=g+2$, $t_1=g-1$ and $t_2=g$. The set $\mathcal{T}$ of Takahashi lengths is given by $\mathcal{T}=(1,2,\ldots,g,g+1,2g+1)$. The set $\tilde{\mathcal{T}}$ of truncated Takahashi lengths is given by $\tilde{\mathcal{T}}=(1,2)$. From [21, §1.11], $\boldsymbol{B}$ is the $(g-1)\times(g-1)$ matrix $\boldsymbol{B}=\{B_{j\ell}\}_{1\leq j,\ell\leq g-1}$ with entries $B_{j\ell}=\min\{j,\ell\}$, and $\overline{\boldsymbol{C}}$ is the 2×2 matrix $\left(\begin{smallmatrix}1 & 1\\ -1 & 2\end{smallmatrix}\right)$. The matrix $\overline{\boldsymbol{C}}^{*}$ is the 2×2 matrix $\left(\begin{smallmatrix}-1 & 2\\ 0 & -1\end{smallmatrix}\right)$.

Throughout this subsection, we take $N_j=n_j+n_{j+1}+\cdots+n_{g-1}$ for $1\leq j\leq g-1$, and the $(g+2)$-dimensional vectors $\boldsymbol{e}_j$ are defined by $\boldsymbol{e}_j=(\delta_{1j},\delta_{2j},\ldots,\delta_{g+2,j})$ for $0\leq j\leq g+3$.

Lemma 3.7. *Let $g\geq 1$ and $1\leq s\leq g+1$. Then:*

$$\chi^{4,4g+3}_{1,s} = \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\equiv 0 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g}{4}m_1^2+\frac{1}{2}m_2^2+\frac{1}{2}(g+1-s)m_1+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix}\frac{1}{2}m_1\\ m_2\end{bmatrix}_q \tag{32a}$$

and

$$\chi^{4,4g+3}_{3,s} = q^{s-2g-\frac{3}{2}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1}\in\mathbb{Z}_{\geq 0}\\ m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1\equiv 0 \,(\mathrm{mod}\, 2)\\ m_2\equiv 1 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1)+\frac{g}{4}m_1^2+\frac{1}{2}m_2^2+\frac{1}{2}(g+1-s)m_1+(N_s+N_{s+1}+\cdots+N_{g-1})}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix}\frac{1}{2}m_1\\ m_2\end{bmatrix}_q. \tag{32b}$$

Each term $(N_s+N_{s+1}+\cdots+N_{g-1})$ is to be taken as 0 whenever $s\geq g$.

Proof: Since with $s\leq g+1$ and $r=1$, we have $s\in\mathcal{T}$ and $r\in\tilde{\mathcal{T}}$, the fermionic expression [21, (1.16)] for $\chi^{p,p'}_{r,s}$ comprises a single fundamental fermionic form [21, (1.17)]. The summation in [21, (1.17)] is then over integers $n_1,n_2,\ldots,n_{g-1}$ and m_1 and m_2 (these latter two parameters are named m_g and m_{g+1} in [21, (1.17)]), with the parity of m_1 and m_2 are restricted.

From [21, §1.7], we obtain:

$$\boldsymbol{u}^L = \begin{cases} 0 & \text{if } s = g+1; \\ \boldsymbol{e}_g & \text{if } s = g; \\ \boldsymbol{e}_{s-1} - \boldsymbol{e}_{g-1} - \boldsymbol{e}_g & \text{if } s < g. \end{cases}$$

From [21, §1.9], we then get $\overline{\boldsymbol{u}}^L_\flat = (0,0)$ if $s = g+1$ and $\overline{\boldsymbol{u}}^L_\flat = (-1,0)$ if $s \le g$. On the other hand, $\boldsymbol{u}^R = 0$ and $\overline{\boldsymbol{u}}^R_\sharp = (0,0)$. For [21, (1.18)], we then get $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{g-1})$ where, if $s \in \{g, g+1\}$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m_1$ and $\tilde{n}_j = n_j$ for $1 \le j < g-1$, and if $1 \le s < g$ then $\tilde{n}_{g-1} = n_{g-1} + \frac{1}{2}m_1 + \frac{1}{2}$, $\tilde{n}_{s-1} = n_{s-1} - \frac{1}{2}$, and $\tilde{n}_j = n_j$ for $1 \le j < s-1$ and $s \le j < g-1$. The expression [21, (1.17)] then yields equation (32a) up to an overall factor, where m_1 and m_2 are each summed over even integers because here $\overline{\boldsymbol{u}} = (0,0)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (0,0)$. The overall factor may be obtained by noting that the smallest value of the exponent in the numerator of the summand in equation (32a) occurs when $n_1 = n_2 = \cdots = n_{g-1} = m_1 = m_2 = 0$. Since $\chi^{p,p'}_{r,s}|_{q=0} = 1$, it follows that the required factor is simply 1.

For the second expression, we obtain $\boldsymbol{u}^L$ as above. Using $r = 3$ instead of $r = 1$ results in $\boldsymbol{u}^R = \boldsymbol{e}_{g+2}$. This yields $\overline{\boldsymbol{u}}^R_\sharp = (0,0)$, as above. However here, $m_1 \equiv 0 \pmod 2$ and $m_2 \equiv 1 \pmod 2$ in the summation because $\overline{\boldsymbol{u}} = (0,1)$ leads to $\overline{\boldsymbol{Q}}(\boldsymbol{u}) \equiv (-2,-1)$. Thus, apart from the overall factor, the fermionic form for $\chi^{4,4g+1}_{3,s}$ differs from the $r = 1$ case above only in that m_2 is summed over odd integers. The smallest value of the exponent in the numerator of the summand in equation (32b) occurs when $n_1 = n_2 = \cdots = n_{2g-1} = 0$ and $m_1 = 2$ and $m_2 = 1$. It follows that the required factor is $q^{s-2g-\frac{3}{2}}$. □

Proof of Theorem 2.7: The required result is obtained from Lemma 3.7 and (29b) along the lines used to prove Theorem 2.5. □

Lemma 3.8. *Let $g \ge 1$. Then:*

$$\chi^{4,4g+3}_{1,2g+1} = q^{-\frac{g}{4}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1} \in \mathbb{Z}_{\ge 0} \\ m_1,m_2 \in \mathbb{Z}_{\ge 0} \\ m_1 \equiv 1 \,(\mathrm{mod}\, 2) \\ m_2 \equiv 0 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1) + \frac{g}{4}m_1^2 + \frac{1}{2}m_2^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1+1) \\ m_2 \end{bmatrix}_q \tag{33a}$$

and

$$\chi^{4,4g+3}_{3,2g+1} = q^{-\frac{g+2}{4}} \sum_{\substack{n_1,n_2,\ldots,n_{g-1} \in \mathbb{Z}_{\ge 0} \\ m_1,m_2 \in \mathbb{Z}_{\ge 0} \\ m_1,m_2 \equiv 1 \,(\mathrm{mod}\, 2)}} \frac{q^{\sum_{j=1}^{g-1} N_j(N_j+m_1) + \frac{g}{4}m_1^2 + \frac{1}{2}m_2^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{g-1}}(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1+1) \\ m_2 \end{bmatrix}_q. \tag{33b}$$

Proof: Here $s = 2g+1$ so that $s \in \mathcal{T}$. Thereupon, $\boldsymbol{u}^L = \boldsymbol{e}_{g+1}$, $\overline{\boldsymbol{u}}^L_\flat = (0,0)$ and $\tilde{\boldsymbol{n}} = (n_1, n_2, \ldots, n_{g-2}, n_{g-1} + \frac{1}{2}m_1)$. The proof now follows the lines of the proof of Lemma 3.7, noting that for $r = 1$, we obtain $\boldsymbol{u} = (1,0)$ whereupon $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (1,0)$; and for $r = 3$, we obtain $\boldsymbol{u} = (1,1)$ whereupon $\overline{\boldsymbol{Q}}(\boldsymbol{u}) = (1,1)$. □

Proof of Theorem 2.8: The required result is obtained from Lemma 3.8 and (29b) along the lines used to prove Theorem 2.6. □

4. Discussion

In this paper, for $p = 3$ and $p = 4$, we have proved fermionic-product identities for sums $\chi^{p,p'}_{1,s} + q^{\Delta}\chi^{p,p'}_{p-1,s}$ of characters for particular choices of Δ.

For $p > 5$, Bytsko and Fring [7] showed that the corresponding sums of characters do not have a product form. Furthermore, a detailed investigation of the fermionic expressions of [21] (which is not reported in this work) shows that it is also not possible to directly sum such expressions to obtain a single fermionic form. Thus these $p > 5$ cases do not lead to analogues of the Andrews-Gordon identities.

In [7], product forms were also derived for the difference of two characters when $p = 3$, $p = 4$ and $p = 6$. In the $p = 3$ cases in which p' and $p'/2 - s$ are both even, these lead to new identities similar to our $M(p, p')^+_2$-identities with a sign factor $(-1)^M$ incorporated into the summands. In the other $p = 3$ cases and all of the $p = 4$ cases, the identities so obtained are equivalent to our $M(p, p')^+_2$-identities.

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